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# FUNCTIONS WHICH MAP THE INTERIOR OF THE UNIT CIRCLE UPON SIMPLE REGIONS.

By J. W. Alexander, II.

## § 1.

A necessary and sufficient condition that a function $w = w(z)$ which is analytic at every point within the circle $| z | = 1$ shall map the interior of the circle upon a simple region in the $w$ plane is that

$$\text{(I)} \qquad \frac{w(z_1) - w(z_2)}{z_1 - z_2} \neq 0$$

as long as $z_1$ and $z_2$ are points within the circle. It is proposed to derive a number of other conditions, sufficient though not necessary, which can often be applied more readily than (I) in actually testing a given function. By means of these, it will be possible to determine certain classes of functions which map the interior of the unit circle upon simple regions such as, for example, the functions (§ 8)

$$\text{(1)} \qquad w = a_1 z + a_2 \frac{z^2}{2} + a_3 \frac{z^3}{3} + \cdots$$

and

$$\text{(2)} \qquad w = a_1 z + \frac{a_3 z^3}{3} + \frac{a_5 z^5}{5} + \cdots$$

where, in both cases, the $a$'s form a non-increasing sequence of positive numbers.

In the earlier part of the paper, the case where the function $w(z)$ is a polynomial will be considered. A study will be made of the relation between the mapping of the unit circle and the position of the roots of $w$ and of $dw/dz$.

## § 2.

A polynomial $w(z)$ of degree $n$ maps the $z$ plane conformally upon an $n$-sheeted Riemann surface $W$, except that at the branch points of $W$, defined by the vanishing of $dw/dz$, angles fail to be preserved. The unit circle in the $z$ plane is mapped upon a curve $C$ whose shape is determined except perhaps for a translation, rotation, and magnification when the position of the roots of $w(z)$ or of $dw/dz$ is known. If we write

$$\text{(3)} \qquad w = cz^k(z - z_{k+1})(z - z_{k+2}) \cdots (z - z_n)$$

where $z_{k+1}$, $z_{k+2}$, $\cdots$, $z_n$ denote the roots of $w$ other than 0, it becomes evident that the change in the argument of $w$ as $z$ varies from $z'$ to $z''$ is equal to the sum of the angles generated at the roots of $w$ by the vectors joining the latter to $z$, each root being counted with its proper multiplicity.*

The ratio between the length of an element of arc in the $w$ plane and the corresponding element in the $z$ plane, or the *distortion* of the map, is $| dw/dz |$; the angle between the two elements, or the *twist*, is arg $dw/dz$. If we write

$$(4) \qquad \frac{dw}{dz} = az^{k-1}(z - \zeta_k)(z - \zeta_{k+1}) \cdots (z - \zeta_{n-1})$$

we may easily visualize the distortion and twist at any point, as well as the way in which the twist varies when $z$ moves from $z'$ to $z''$.

As an immediate consequence of the above, it is clear that if all the roots of the polynomial $w$ are on the same side of a straight line, the line is transformed into a curve which turns continuously about the origin, while if all the roots of the derived function are on the same side of a line, the line is transformed into a curve whose curvature is either everywhere positive or everywhere negative.

We shall bring these preliminaries to a close by recalling that the roots of $w$ have the same arithmetic mean as the roots of $dw/dz$, for if

$$w = a_0 + \cdots + a_{n-1}z^{n-1} + a_n z^n,$$

both arithmetic means fall at $-\dfrac{a_{n-1}}{na_n}$.

## § 3.

Omitting the trivial case where the degree $n$ of the polynomial $w$ is unity, let us determine in the $z$ plane a region $R_n$ such that every polynomial $w$ whose roots are all within or on the boundary of $R_n$ shall map the unit circle upon a simple region. By observing for what values of $a$ the function

$$(5) \qquad w = (a - z)^n$$

fails to have this property, we can immediately set bounds to the region $R_n$. Thus, points within the unit circle are excluded at once, since $dw/dz$ also vanishes at $a$. Moreover, since the function (5) transforms circles about $a$ into circles about the origin and multiplies by $n$ the angle subtended by an arc of the first of these circles at its center, the point $a$ cannot lie so close

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* This geometrical method of interpretation may, of course, be extended to the case of a function defined by a convergent infinite product

$$(3') \qquad w = cz^k(1 - z/z_{k+1})^{l_1}(1 - z/z_{k+2})^{l_2} \cdots,$$

where the $l$'s need only be real. The angle generated at $z_{k+2}$, must then be multiplied by $l_i$.

to the unit circle that the two tangent rays from $a$ enclose an angle of more than $2\pi/n$. Therefore, *the region $R_n$ contains no point within the circle $K$ of radius $1/\sin(\pi/n)$ about the origin.*

Now if we draw any tangent $T$ to the unit circle as, for example, the vertical tangent through the point 1, we may take for the region $R_n$ that portion of the half-plane to the right of $T$ which lies without the circle $K$. For when the roots of the polynomial $w$ are all *within* the region $R_n$ as thus defined, the roots of $dw/dz$ are all to the right of the tangent $T$, by the well-known theorem* that the roots of the derivative of a polynomial are all contained within every convex polygon which encloses all the roots of the polynomial itself. The left half-plane bounded by $T$ is therefore mapped upon a region containing none of the branch points of $W$ and bounded by the image of $T$, a curve which winds about the origin continually in the same sense. As a consequence of this, two points $z'$ and $z''$ of the left half-plane correspond to the same point $w$ only if the linear segment $z'z''$ is mapped upon a curve which encloses the origin. But this is impossible when $z'$ and $z''$ are within or on the unit circle, for they then subtend an angle of less than $2\pi/n$ at each of the roots of $w$, or in all an angle less than $2\pi$. If the roots of $w$ are on the boundary of $R_n$, the image of the unit circle may touch itself, but the interior will still be mapped upon a simple region. The region $R_n$ therefore has the required property.

We shall now prove that the region $R_n$ cannot be enlarged by the addition of other points of the plane. Insofar as the points within the circle $K$ of radius $1/\sin(\pi/n)$ are concerned, this has already been proved; there remain to be considered the points outside or on this circle and to the left of the tangent $T$. Every point $P$ of the latter class may be joined to a point $Q$ within the unit circle so near to the point 1 that when $PQ$ is produced beyond $Q$ to a point $Q'$ such that $\overline{PQ'} = n\overline{PQ}$, $Q'$ shall lie within $R_n$. If we then consider the polynomial $w$ which vanishes $n-1$ times at $Q'$ and once at $P$, we find that since $dw/dz$ vanishes $n-2$ times at $Q'$ it must vanish the remaining time at $Q$ within the unit circle, by the theorem that the roots of $w = 0$ and of $dw/dz = 0$ have the same arithmetic mean. The point $P$ therefore cannot be added to the region $R_n$, and we have

CONDITION (II). *If the roots of a polynomial $w(z)$ of the nth degree all lie within or on the boundary of a region $R_n$ consisting of the points outside a circle of radius $1/\sin(\pi/n)$ about the origin and on the side of a tangent to the unit circle away from the circle, the interior of the circle is mapped upon a simple region. Moreover, the test no longer holds if the region $R_n$ be enlarged.*

A function

$$w = (z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}\cdots(z-a_k)^{\alpha_k}, \tag{6}$$
$$\alpha_i > 0, \qquad i = 1, 2, \cdots, k, \qquad \Sigma\alpha_i = n$$

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* A simple proof of this theorem is given by F. Irwin, these *Annals*, vol. 16 (1915), p. 138.

may be regarded as a *generalized polynomial* of degree $n$, where $n$ need not be an integer. It will then be found that Condition (II) holds true for generalized as well as ordinary polynomials. In fact, the entire argument, including the proof of the theorem referred to in the footnote on p. 14, may be applied with only trivial modifications to these more general functions. The argument may also be extended to many functions defined by infinite products.

## § 4.

A region every point of which may be joined to a point $a$ by means of a linear segment consisting only of points of the region will be called a *star-shaped region* with center at $a$. When an arbitrary point of the region may be chosen as center, the region will be called *convex.*

Now, let $w = w(z)$ map the unit circle upon a star-shaped region. Then arg $w$ is a never decreasing function of $\theta = \arg z$ as $z$ describes the unit circle in the positive sense. If $w = w(z)$ maps the unit circle upon a convex region, the slope, $\arg z(dw/dz)$, of the image of a radius of the circle is a never decreasing function of $\theta$ as $z$ describes the unit circle, or as $z$ describes the circle $|z|/< 1 - \epsilon$ when $\arg z(dw/dz)$ fails to exist on the unit circle. We can therefore state the

THEOREM. *A necessary and sufficient condition that the function $w = w(z)$ map the interior of unit circle upon a convex region is that $z(dw/dz)$ shall map the interior of the circle upon a star-shaped region with center at the origin.*

Let us now determine the largest region $S_n$ such that if the $n - 1$ roots of the polynomial

$$w = zP_{n-1}(z) \tag{7}$$

other than zero lie within or upon the boundary of $S_n$, the image of the interior of the unit circle shall be a star-shaped region.

This problem may be solved at once. Clearly, the region $S_n$ contains no point $a$ within a circle of radius $n$ about the origin, for the derivative of the function

$$w = z(z - a)^{n-1} \tag{8}$$

vanishes $n - 2$ times at $a$ and once at $a/n$ so that we must have $|a| \geqq n$. On the other hand, if the remaining $n - 1$ roots of $w$ are all without a circle of radius $n$, the angle subtended at each by an element of arc $ds$ on the unit circle is less than $ds/(n - 1)$, therefore the sum of the angles subtended at them all is less than $ds$, the angle subtended at the root $z = 0$. The unit circle is therefore mapped upon a curve which turns about the origin continually in the same sense, and its interior is therefore mapped upon a star-shaped region. We therefore have the

THEOREM (CONDITION III). *The function* $w = zP(z)$, *where* $P(z)$ *is a polynomial of the* $(n - 1)$*st degree maps the interior of the unit circle upon a star-shaped region with center at the origin if the roots of* $P(z)$ *all lie without or upon a circle of radius n about the origin. Moreover, the region* $S_n$ *composed of the points without or on the circle cannot be enlarged by the addition of other points of the plane.*

Regarding $zP(z)$ as the function $z(dw/dz)$ in the first theorem of this section, we have the

COROLLARY. *The polynomial* $w = w(z)$ *of the nth degree maps the interior of the unit circle upon a convex region if the roots of* $(dw/dz) = 0$ *all lie without or upon a circle of radius n about the origin. Moreover, the region thus defined cannot be enlarged.*

We also have the

COROLLARY. *The polynomial* $w = w(z)$ *of the nth degree maps the interior of the unit circle upon a convex region if its roots all lie within or on the boundary of the half-plane which is bounded by a line at a distance n from the origin and which does not contain the origin.*

For the roots of $(dw/dz) = 0$ then all lie without the circle of radius $n$ about the origin.

COROLLARY. *Let* $z_1$ *be a simple root of the polynomial* $w(z)$ *of the nth degree and let d be the distance from* $z_1$ *to the nearest other root of* $w(z)$. *Then* $dw/dz$ *cannot vanish within a circle of radius* $d/n$ *about* $z_1$. *Moreover, there exists a polynomial satisfying the conditions of the corollary and such that* $dw/dz$ *vanishes on the circle of radius* $d/n$.

For, by a linear transformation, $z_1$ may be transformed into the origin and the circle of radius $d/n$ into the unit circle.

The extension of the results of this section to the case of generalized polynomials suggests itself at once and need not be insisted upon.

## § 5.

The position of the roots of $dw/dz$ also determines whether or not the interior of the unit circle is mapped by $w = w(z)$ upon a non-overlapping region, hence it ought to be possible to find the greatest region $R_n'$ such that when the roots of $dw/dz$ all lie within $R_n'$, the function $w(z)$ satisfies Condition (I). This problem appears, however, to be much more difficult than the analogous one considered in § 3. We shall state a few theorems about the region $R_n'$ without defining it completely.

*The region* $R_n'$ *contains no point within a circle of radius* $1/\sin(\pi/n)$, as may be seen at once by going back to Function (5). On the other hand, a circle about the origin can easily be found outside of which there are only points of $R_n'$. For when all the roots of $dw/dz$ lie without the unit circle, the image $C$ of the latter can overlap only if, when regarded as a curve of the $w$ plane, it contains a closed loop winding in the negative sense about one or more branch points of the Riemann surface $W$. The presence of this loop is evident if we consider the variation in the image of a circle $K$ concentric with the unit circle as the radius of $K$ increases from 0 to 1. A double point can appear on the image of $K$ only if the curve passes through a position in which it has a cusp or if it makes one of the negative loops in question. Moreover, a negative loop having once appeared cannot disappear unless the curve goes through a position in which it has a cusp. Now the curve cannot have a cusp at any stage since $dw/dz$ does not vanish within the unit circle, therefore, if the curve crosses itself, it must include a negative loop. But the difference between the distortion at the beginning of the loop and that at the end must then be greater than $\pi$ plus the angle subtended by the arc of the unit circle corresponding to the loop. The curve $C$ therefore cannot cross itself if

$$\text{(IV)} \qquad \begin{gathered} \frac{dw}{dz} \neq 0 \text{ within the unit circle and} \\ \arg\left(\frac{dw}{dz}\right)_1 - \arg\left(\frac{dw}{dz}\right)_2 \leqq \pi + \arg(z_2 - z_1) \end{gathered}$$

where $z_1z_2$ is a positive arc on the unit circle. The second inequality may be written

$$\text{(IV}'\text{)} \qquad \arg \frac{\left(z\dfrac{dw}{dz}\right)_1}{\left(z\dfrac{dw}{dz}\right)_2} \leqq \pi.$$

*A fortiori*, the curve $C$ does not cross itself if

$$\text{(V)} \qquad \arg \frac{\left(\dfrac{dw}{dz}\right)_1}{\left(\dfrac{dw}{dz}\right)_2} \leqq \pi.$$

Therefore, we have

CONDITION (VI). *If the sum of the angles subtended at the roots of $dw/dz$ by the tangents from these points to the unit circle is less than or equal to $\pi$, the interior of the unit circle is mapped by the function $w = w(z)$ upon a simple region.*

It also follows that *the region $R_n'$ contains all the points without or upon the circle of radius $1/\sin(\pi/2(n-1))$ about the origin.*

A better approximation of the region $R_n'$ could of course be obtained by using Condition (IV) rather than (V), but (IV) does not appear to give the entire region either. If we go back to the discussion in § 2, we see that the twist at the point of maximum twist on the unit circle tends to increase as the roots of $dw/dz$ approach the unit circle and that it tends to be greater when the roots are grouped together at a point rather than evenly distributed around the circle. It is therefore highly probable that the region $R_n'$ is determined by Function (5) and consists of the points outside the circle of radius $1/\sin(\pi/n)$ about the origin as center.

## § 6.

Condition (V) implies that the range of variation of $dw/dz$ on the unit circle is less than $\pi$, or in other words that the function $w' = (dw(z)/dz)$ maps the interior of the circle upon a region contained within a half-plane bounded by a straight line through the origin. The problem of finding functions $w'$ which satisfy Condition (V) therefore reduces to one already examined by Carathéodory. If $w'$ be expanded about the origin in a series

$$w' = a_1 + a_2 z + a_3 z^2 + \cdots, \tag{9}$$

a necessary and sufficient condition that $w'$ map the interior of the unit circle upon a region contained in a given half plane is that the real and imaginary parts of the first $n$ coefficients $a_1, a_2, a_3, \cdots$ shall for every $n$ be the homogeneous coordinates of a point in a certain convex domain in $2n$ space.* The $a$'s may be made to appear in the series defining $w$ by writing

$$w = A + a_1 z + \frac{a_2 z^2}{2} + \cdots + \frac{a_n z^n}{n} + \cdots. \tag{10}$$

As a result of Condition V, we have the

THEOREM. *If in the series (10),* $|a_1| \geqq |a_2| + |a_3| + \cdots + |a_n| + \cdots$, *the interior of the unit circle is mapped upon a simple region.*

For $dw/dz$ always lies in the half-plane bounded by the normal to the vector $a_1$ through the origin and containing the point $a_1$.

In deriving Condition (V), we really assumed that $w$ was a polynomial, but there is no difficulty in showing that (V) holds equally well for a convergent infinite series. That no two points within the unit circle can be mapped upon the same point $w$ will then be seen by enclosing them within a circle interior to and concentric with the unit circle. The image of the interior of the latter circle can be approximated as closely as we please by a series of a finite number of terms.

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* C. Carathéodory, "Über den Variabilitätsbereich der Koeffizienten, etc.," Math. Ann., vol. LXIV (1907), p. 95.

## § 7.

Although no root of $dw/dz$ can lie within the unit circle if the curve $C$ is to bound a simple region, it is possible for all the roots to lie upon the circle itself provided they are not too close to one another. Let us suppose that they are all upon the unit circle and follow one another in the order $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ as the circle is described in the positive sense. Then every arc $\zeta_i\zeta_{i+1}$ is mapped upon a curve of positive curvature, since a point in describing the arc turns in a positive sense about all the points $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$. The curve $C$ therefore consists of $n-1$ arcs all concave inward and abutting in pairs at the branch points of the Riemann surface $W$ at each of which they form a cusp of $C$. It may further be seen that the curve $C$ cannot cross itself if the angle made by the tangent to $C$ increases by at least $\pi$ as we travel along the curve from one cusp to the next. Because, under these conditions, the curve $C$ would only cross itself by containing a simple loop winding in the positive sense about the region which it enclosed. But if the curve $C$ contained such a loop, so also would the variable curve $C'$ defined as the image of a circle concentric with the unit circle but of radius less than 1. For as we varied the radius from 1 to 0, the loop could only disappear by shrinking to a cusp, which is impossible. On the other hand, we know that when the radius is sufficiently small, the curve $C'$ differs by as little as we please from a circle.

Now the angle through which the tangent to $C$ turns is equal to the angle $\theta$ subtended by the corresponding arc of the unit circle at the center of the circle plus the sum of the angles subtended at the points $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ or

$$\theta + \frac{n-1}{2}\theta = \frac{n+1}{2}\theta.$$

The curve $C$ therefore cannot cross itself if

$$\text{(VII)} \qquad \theta \geqq \frac{2\pi}{n+1}.$$

By applying this test, we see that the functions

$$\text{(11)} \qquad w = z - \frac{z^n}{n},$$

$$\text{(12)} \qquad w = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n},$$

$$\text{(13)} \qquad w = z + \frac{z^3}{3} + \frac{z^5}{5} + \cdots + \frac{z^{2k+1}}{2k+1},$$

map the interior of the unit circle upon simple regions since their derivatives are

$$(11') \qquad \frac{dw}{dz} = 1 - z^{n-1}, \qquad \theta = \frac{2\pi}{n-1},$$

$$(12') \qquad \frac{dw}{dz} = 1 + z + \cdots + z^{n-1}, \qquad \theta = \frac{2\pi}{n},$$

$$(13') \qquad \frac{dw}{dz} = 1 + z^2 + \cdots + z^{2k}, \qquad \theta = \frac{2\pi}{n+1} \quad (n = 2k+1);$$

respectively. We shall have occasion to consider the above functions presently.

## § 8.

We shall mention one more test leading to a number of applications. The interior of the unit circle will be mapped upon a simple region if it is possible to draw from every point $w(e^{i\theta})$ of the curve $C$ a ray which satisfies the following three conditions:

(*a*) The angle through which it must be turned in order to make its direction coincide with the positive direction along the tangent to the curve $C$ shall be between 0 and $\pi$ inclusive.

(*b*) The angle $\psi$ which it makes with the positive axis of reals shall be a never-decreasing function of $\theta$ ($= \arg z$).

(*c*) The angle $\psi$ shall not increase by more than $2\pi$ as $\theta$ varies from 0 to $2\pi$.

Condition (*a*) may be expressed thus:

$$\pi \geqq \arg\left(z\frac{dw}{dz}\right) + \frac{\pi}{z} - \psi \geqq 0$$

or

$$(14) \qquad \frac{\pi}{2} \geqq \arg\left(z\frac{dw}{dz}\right) - \psi \geqq -\frac{\pi}{2}.$$

It means geometrically that the ray never points to the side of $C$ corresponding to the interior of the unit circle.

To prove that these three conditions are sufficient, we observe first that $dw/dz$ cannot vanish within the unit circle, otherwise the tangent to the curve $C$ would turn through an angle $2k\pi$, ($k > 1$) as $\theta$ varied from 0 to $2\pi$. It would then be impossible to find a set of rays satisfying the first and last conditions simultaneously. On the other hand, when the roots of $dw/dz$ all lie without or on the circle, the curve $C$ cannot cross itself unless it contains a closed loop winding in the negative sense about the region which it encloses (cf. § 5). The slope of the tangent then decreases by at least $\pi$ as the loop is described and Conditions (*a*) and (*b*) cannot both be satisfied.

Let us apply this test to the function

$$w_n = z + \frac{z^2}{2} + \cdots + \frac{z^n}{n} \cdots \tag{12''}$$

previously considered. If we put

$$\psi = \arg \frac{z}{1 - z},$$

conditions (*b*) and (*c*) are certainly satisfied. Moreover,

$$\arg z \frac{dw_n}{dz} - \arg \frac{z}{1 - z} = \arg (1 - z^n),$$

which satisfies (14) and therefore Condition (*a*). We are enabled, however, to go much further than before if we observe that $\psi$ is independent of $n$, the degree of the polynomial $w_n$. For if two functions $w_i$ and $w_j$ satisfy (14) for the same choice of $\psi$, so also does their sum, since $\arg (d/dz)(w_i + w_j)$ lies somewhere between $\arg (dw_i/dz)$ and $\arg (dw_j/dz)$. More generally, if $c_1, c_2, \cdots, c_n$ be a set of positive constants and $w_1, w_2, \cdots, w_n$ a set of functions satisfying (14) for the same choice of $\psi$, the sum function

$$w = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n \tag{15}$$

also satisfies (14).

THEOREM. *The function*

$$w = a_1 z + \frac{a_2 z^2}{2} + \cdots + \frac{a_n z^n}{n},$$

*where the a's form a non-increasing sequence of positive numbers maps the interior of the unit circle upon a simple region.*

For $w$ may be written as

$$w = a_n w_n + (a_{n-1} - a_n) w_{n-1} + \cdots + (a_1 - a_2) w_1$$

where $w_n$ is the function defined in (12''). The function $w$ is thus seen to be of the form (15).

Starting from

$$w = z + \frac{z^3}{3} + \cdots + \frac{z^n}{n}, \qquad n = 2k + 1 \tag{13}$$

and putting $\psi = \frac{1 + z}{1 - z}$, we have

$$\arg z \frac{dw}{dz} - \arg \frac{1 + z}{1 - z} = \arg z \frac{(1 - z^n)}{(1 + z)^2} = \arg (1 - z^n)$$

which also satisfies (14). This leads us to the

THEOREM. *The function*

$$w = a_1 z + \frac{a_3 z^3}{3} + \cdots + \frac{a_{2k+1} z^{2k+1}}{2k+1},$$

*where the a's form a non-increasing sequence of positive numbers maps the interior of the unit circle upon a simple region.*

Other analogous theorems will easily suggest themselves. By the method indicated at the end of § 6, we may pass to the case of infinite series and obtain the theorems mentioned in the introduction.

## § 9.

By the method of § 8 and starting from the function (11), we could have proved the theorem of § 6. We shall now prove the stronger

THEOREM. *The function*

$$w = a_1 z + \frac{a_2 z^2}{2} + \cdots + \frac{a_n z^n}{n} + \cdots,$$

*where* $| a_1 | \geqq \sum_{i=2}^{\infty} | a_i |$ *maps the interior of the unit circle upon a star-shaped region with center at 0.*

To prove this theorem, it will be sufficient to show that arg $w$ is a never decreasing function of $\theta = \arg z$. But

$$\begin{aligned} \frac{d}{d\theta} \arg w &\geqq | a_1 | \left| \frac{d}{d\theta} \arg z \right| - \sum_{i=2}^{\infty} | a_i | \left| \frac{d}{d\theta} \arg \frac{z^i}{i} \right| \\ &\geqq | a_1 | - \sum_{i=2}^{\infty} | a_i | \\ &\geqq 0, \end{aligned}$$

which proves the theorem.

As a corollary, we have the

THEOREM. *The function*

$$w = a_1 z + \frac{a_2 z^2}{2^2} + \cdots + \frac{a_n z^n}{n^2} + \cdots,$$

*where* $| a_1 | \geqq \sum_{i=2}^{\infty} | a_i |$ *maps the interior of the unit circle upon a convex region.*

For $z(dw/dz)$ maps the circle upon a star-shaped region with center at 0.

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